Inequality

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Prove that if $x,y,z>0$, then

$$\frac{4(x^2+y^2+z^2)}{27(xy+yz+zx)}+\sum\frac{x}{7x+y+z}\geq\frac{13}{27}.$$

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We will prove inequality of the problem for any $x,y,z\geq 0$ such that $xy+yz+zx>0$.
Due to homogeneity of inequality assume $x+y+z=1$. Also denote $xy+yz+zx$ and xyz respectively, via p and q. Then $x^2+y^2+z^2=1-2p, \sum\frac{x}{7x+y+z}=\sum\frac{x}{1+6x}=$
$\frac{\sum x((1+6y)(1+6z))}{(1+6x)(1+6y)(1+6z)}=\frac{1+12p+108q}{7+36p+216q}$ and inequality of the problems becomes

(**1**) $$\frac{4(1-2p)}{27p}+\frac{1+12p+108q}{7+36p+216q}\geq\frac{13}{27}.$$

Note that $0<p\leq 1/3$ ($3(xy+yz+zx)\leq(x+y+z)^2$) and $9q\geq\max\{0,4p-1\}$
($9q\geq 4p-1$ is normalized by $x+y+z=1$ form of Schure Inequality $\sum x(x-y)(x-z)\geq 0$ in p,q-notation). Since $\frac{1+12p+108q}{7+36p+216q}=\frac{1}{2}-\frac{5+12p}{2(7+36p+216q)}$ increasing in $q\geq 0$ then:

1. For $p\in[1/4,1/3]$ we obtain

$$\frac{4(1-2p)}{27p}+\frac{1+12p+108q}{7+36p+216q}-\frac{13}{27}\geq\frac{4(1-2p)}{27p}-\frac{13}{27}+\frac{1+12p+12(4p-1)}{7+36p+24(4p-1)}=$$
$$\frac{4-21p}{27p}+\frac{11-60p}{17-132p}=\frac{4(1-3p)}{27p}\cdot\frac{96p-17}{132p-17}\geq 0 \text{ because}$$
$$\frac{96p-17}{132p-17}\geq\frac{96\cdot\frac{1}{4}-17}{132\cdot\frac{1}{4}-17}=\frac{7}{16}>0 \; (\frac{96p-17}{132p-17} \text{ increasing in } p>0);$$

2. For $p\in(0,1/4)$ we obtain

$$\frac{4(1-2p)}{27p}+\frac{1+12p+108q}{7+36p+216q}-\frac{13}{27}\geq\frac{4(1-2p)}{27p}+\frac{1+12p}{7+36p}-\frac{13}{27}=$$
$$\frac{4(7+6p-108p^2)}{27p(36p+7)}\geq\frac{4\left(7+6p-108\cdot(1/4)^2\right)}{27p(36p+7)}=\frac{4(1/4+6p)}{27p(36p+7)}>0.$$